PMM U.S.S.R.,Vol.51,No.3,pp.303-307,1987
Printed in Great Britain

0021-8928/87 $10.00+0.00

CONTROL OF THE MOTION OF A PENDULUM IN A GAME-THEORETIC SITUATION*

V.V. OSTAPENKO

The description of the structure of differential approach-evasion games has, in general, a non-constructive character /1-5/. In order to overcome this drawback separate classes of games, admitting of a relatively simple construction of the players' strategies, were investigated in /6-10/. A method developed in /9, 10/ is applied to a new class of games, including games whose dynamics are described by the pendulum equation.

1. Let X be a finite-dimensional Euclidean space and $C(\tau)$, $\tau \in [0, t]$ an integrable family of linear operators acting from X into X. We shall use the following definition below /11/.

Definition. Let $H \subset \{x^* \in X : \|x^*\| = 1\}$. The set M will be called H-convex if it can be represented in the form

$$\bigcap_{x^* \in H} \{x \in X : \langle x, x^* \rangle \leqslant c(x^*)\} \tag{1.1}$$

where $\langle \cdot, \cdot \rangle$ denotes a scalar product in X and the quantities $c(x^*)$ can take the value $+\infty$.

Let $\omega = \{\tau_0 = 0 < \tau_1 < \ldots < \tau_n = t\}$ be a finite decomposition of the interval $[0, t]$. We denote by H^ω a set of unit vectors $x^* \in X$, not necessarily all vectors, for which the following conditions hold:

a) $C^*(\tau)x^* = \lambda(\tau \mid x^*)x^*$ for which $\tau \in [0, t]$;

b) The scalar function $\lambda(\cdot \mid x^*)$ does not, for any x^*, change its sign in every interval $[\tau_{i-1}, \tau_i)$ $(i = 1, \ldots, n)$.

Let us write

$$C_i = \int_{\tau_{i-1}}^{\tau_i} C(\tau)\, d\tau, \quad i = 1, \ldots, n$$

and assume that the operators C_i have their inverses.

Theorem 1. Let M be a H^ω-convex set, and $x_i(s_1, \ldots, s_i)$, $s_j \in [\tau_{j-1}, \tau_j)$ $(j = 1, \ldots, i;\ i = 1, \ldots, n)$ the functions integrable over the variable manifold, with values in X. Then, if for any $s_i \in [\tau_{i-1}, \tau_i)$ $(i = 1, \ldots, n)$ the following inclusion holds:

$$\sum_{i=1}^{n} C_i x_i(s_1, \ldots, s_i) \in M, \quad \text{then} \quad \int_0^t C(\tau)\, x(\tau)\, d\tau \in M$$

where

$$x(s) = \int_{\tau_{i-2}}^{\tau_{i-1}} C_{i-1}^{-1} C(\omega_{i-1}) \ldots \int_{\tau_0}^{\tau_1} C_1^{-1} C(\omega_1)\, x_i(\omega_1, \ldots, \omega_{i-1}, s)\, d\omega_1, \ldots, d\omega_{i-1}$$

when $s \in [\tau_{i-1}, \tau_i]$.

Proof. We have, by virtue of the convex and closed nature of M,

$$\sum_{i=1}^{n} \int_{\tau_{i-1}}^{\tau_i} C(\tau)\, d\tau W_i \subset M \tag{1.2}$$
$$W_i = \overline{\text{conv}}\, \{x_i(s_1, \ldots, s_i), \quad s_j \in [\tau_{j-1}, \tau_j)\}$$

*Prikl.Matem.Mekhan.,51,3,390-394,1987

Let M be representable in the form (1.1) where $H = H^{\omega}$. The following sequence of equations holds for any $x^* \in H^{\omega}$ (the summation is carried out over i from 1 to n):

$$\Big\langle \sum \int_{\tau_{i-1}}^{\tau_i} C(\omega_i) \int_{\tau_{i-2}}^{\tau_{i-1}} C_{i-1}^{-1} C(\omega_{i-1}) \ldots \int_{\tau_0}^{\tau_1} C_1^{-1} C(\omega_1) x_i \, d\omega_1 \ldots$$

$$d\omega_i, x^* \Big\rangle = \sum \int_{\tau_{i-1}}^{\tau_i} \Big\langle \int_{\tau_{i-2}}^{\tau_{i-1}} C_{i-1}^{-1} C(\omega_{i-1}) \ldots \int_{\tau_0}^{\tau_1} C_1^{-1} C(\omega_1) \times$$

$$x_i \, d\omega_1 \ldots d\omega_{i-1}, \quad C^*(\omega_i) x^* \Big\rangle d\omega_i =$$

$$\sum \int_{\tau_{i-1}}^{\tau_i} \lambda(\omega_i) \Big\langle \int_{\tau_{i-2}}^{\tau_{i-1}} C_{i-1}^{-1} C(\omega_{i-1}) \ldots \int_{\tau_0}^{\tau_1} C_1^{-1} C(\omega_1) \times$$

$$x_i \, d\omega_1 \ldots d\omega_{i-1}, \quad x^* \Big\rangle d\omega_i = \ldots$$

$$= \sum \Big\langle \int_{\tau_{i-1}}^{\tau_i} \lambda(\omega_i) \int_{\tau_{i-2}}^{\tau_{i-1}} \lambda_{i-1}^{-1} \lambda_{i-1}(\omega_{i-1}) \ldots \int_{\tau_0}^{\tau_1} \lambda_1^{-1} \lambda(\omega_1) \times$$

$$x_i \, d\omega_1 \ldots d\omega_i, \quad x^* \Big\rangle = \Big\langle \sum \int_{\tau_{i-1}}^{\tau_i} \lambda(\omega_i) \, d\omega_i \bar{x}_i x^* \Big\rangle =$$

$$\Big\langle \sum \int_{\tau_{i-1}}^{\tau_i} C(\omega) \, d\omega \bar{x}_i, x^* \Big\rangle$$

$$(x_i = x_i(\omega_1, \ldots, \omega_i), \quad \lambda(\omega) = \lambda(\omega \mid x^*),$$

$$\lambda_i = \int_{\tau_{i-1}}^{\tau_i} \lambda(\omega) \, d\omega)$$

where $\bar{x}_i \in W_i$ is a suitable point. From (1.2) it follows that the last expression in the above sequence of equations is not greater than $c(x^*)$, and this completes the proof of the theorem.

2. Let us consider a game with a fixed time of termination. Let Z, L be the Euclidean spaces $\dim L \leqslant \dim Z$; $A : Z \to Z$, $\varphi : L \to Z$, $\pi : Z \to L$ the linear operators, U and V compacta in Euclidean spaces and $B : U \times V \to L$ a continuous mapping. The dynamics of the game is described by the following equation:

$$z^{\cdot} = Az + \varphi B(u, v), \; z \in Z, \; u \in U, \; v \in V$$

We write the expression for the terminal set in the form $M_L = \{z \in Z : \pi z \in M\}$, where $M \subset L$ is a closed set. The measurable functions $u(\tau)$ and $v(\tau)$ with values in U and V respectively represent the admissible controls of the players P and E. The aim of the player P is to achieve the inclusion $\pi z(t) \in M$, and that of the player E is to prevent it.

Let us write $X = L$, $C(\tau) = \pi e^{A(t-\tau)} \varphi$ and assume that H^{ω} is the set defined in Sect.1. We write

$$P_{\omega}{}^* M = \bigcap_{v_1 \in V} \bigcup_{u_1 \in U} \cdots \bigcap_{v_n \in V} \bigcup_{u_n \in U} \{z \in Z : \pi e^{At} z + \sum_{i=1}^{n} C_i B(u_i, v_i) \in M\}$$

and assume that C_i^{-1} $(i = 1, \ldots, n)$ exist.

Let $u_i(v_1, \ldots, v_i)$ be a mapping acting from V^i into U such, that for any admissible controls $v_i(s_i)$, $s_i \in [\tau_{i-1}, \tau_i)$ the function $u_i(v_1(s_1), \ldots, v_i(s_i))$ is measurable over the variable manifold.

Let us put

$$x(v_1(\cdot), \ldots, v_{i-1}(\cdot) \mid v_i) = \int_{\tau_{i-2}}^{\tau_{i-1}} C_{i-1}^{-1} C(s_{i-1}) \ldots$$

$$\int_{\tau_0}^{\tau_1} C_1^{-1}(s_1) B(u_i(v_i(s_1), \ldots, v_{i-1}(s_{i-1}), v_i), v_i) \, ds_1 \ldots ds_{i-1}$$

Using the methods of /10/ we can show that if $B(U, v) - H^{\omega}$- is a convex set for all $v \in V$, then the equation

$$B(u, v_i) = x(v_1(\cdot), \ldots, v_{i-1}(\cdot) \mid v_i)$$

will have a solution $u_i^*(v_1(\cdot), \ldots, v_{i-1}(\cdot) \mid v_i) \in U$. We can have several such solutions. Let us assume that u_i^* is the smallest of these solutions in the lexicographic sense. According to Fubini's theorem the function $x(v_1(\cdot), \ldots, v_{i-1}|(\cdot)\,|\,v_i(s_i))$ will be measurable in $s \in [\tau_{i-1}, \tau_i)$, as long as $v_i(s_i)$ is an admissible control of the player *E*. According to Filippov's lemma the function $u_i^*(v_1(\cdot), \ldots, v_{i-1}(\cdot)\,|\,v_i(s_i))$ is also measurable in $s_i \in [\tau_{i-1}, \tau_i)$ and can be used as the control of the player *P* in this interval.

Theorem 2. Let *M* and $B(U, v)$ be convex sets for any $v \in V - H^{\omega}$ and $z_0 \in P_\omega^* M$. A mapping $u_i: V^i \to U$ $(i = 1, \ldots, n)$, exists such that the following assertions hold for any admissible control $v(\tau)$ of the player *E*.

a) the functions $u_i(v_1(s_1), \ldots, v_i(s_i))$ are measurable over the manifold of the variables for $s_i \in [\tau_{j-1}, \tau_j)$ $(j = 1, \ldots, i)$ where $v_j(s_j) = v(s_j)$, $s_j \in [\tau_{j-1}, \tau_j)$ $i = 1, \ldots, n)$;

b) if u_i^* are the mappings constructed above with help of the mappings u_i, then the relation $\pi z(t) \in M$ holds for the trajectory $z(\tau)$ with origin at z_0, corresponding to the controls $u(\tau) = u_i^*(v_1(\cdot), \ldots, v_{i-1}(\cdot)\,|\,v_i(\tau))$, $\tau \in [\tau_{i-1}, \tau_i)$ and $v(\tau)$.

Proof. Let $z_0 \in P_\omega^* M$. Then mappings $u_i(v_1, \ldots, v_i)$, exist such that for any $v_i \in V$

$$\pi e^{At} z_0 + \sum_{i=1}^{n} C_i B(u_i(v_1, \ldots, v_i), v_i) \in M \tag{2.1}$$

We shall show that the mappings $u_i(v_1, \ldots, v_i)$ can be constructed so that the function $u_i(v_1(s_1), \ldots, v_i(s_i))$ will be measurable over the manifold of variables $s_j \in [\tau_{j-1}, \tau_j)$ as long as $v_j(s_j)$ is an admissible control of the player *E*.

Let us write

$$M_k = \bigcap_{v_k \in V} \bigcup_{u_k \in U} \cdots \bigcap_{v_n \in V} \bigcup_{u_n \in U} \left[M - \sum_{i=k}^{n} C_i B(u_i, v_i)\right]$$

The mapping $u_1(v_1)$ can be constructed as the smallest solution, in the lexicographic sense, of the following inclusion:

$$\pi e^{At} z_0 + C_1 B(u_1(v_1), v_1) \in M_2$$

According to Filippov's lemma, the function $u_1(v_1(s_1))$ will be measurable as long as $v_1(s_1)$, $s_1 \in [\tau_0, \tau_1)$ is measurable.

We shall assume that a mapping $u_{k-1}(v_1, \ldots, v_{k-1})$ has been constructed. We can choose $u_k(v_1, \ldots, v_k)$ as the smallest solution, in the lexicographic sense, of the inclusion

$$\pi e^{At} z_0 + \sum_{i=1}^{n} C_i B(u_i(v_1, \ldots, v_i), v_i) \in M_{k+1}$$

According to Filippov's lemma, $u_k(v_1(s_1), \ldots, v_k(s_k))$ is a function measurable in $s_j \in [\tau_{j-1}, \tau_j)$ $(j = 1, \ldots, k)$.

We shall assume that the player *E* has chosen the control $v(\tau)$ and $v_i(\tau) = v(\tau)$, $\tau \in [\tau_{i-1}, \tau_i)$ in the interval $[0, t]$. From (2.1) and Theorem 1 it follows that

$$\pi e^{At} z_0 + \sum_{i=1}^{n} \int_{\tau_{i-1}}^{\tau_i} C(\tau) B(u_i^*(v_1(\cdot), \ldots, v_{i-1}(\cdot)\,|\,v_i(\tau)), \quad v_i(\tau))\, d\tau \in M$$

which completes the proof of the theorem.

Theorem 3. Let $B(U, v) - H^{\omega}$ be a convex set for any $v \in V$ and $z_0 \overline{\in} P_\omega^* M$. Then there exist mappings $v_i^*(u_1(\cdot), \ldots, u_{i-1}(\cdot))$, $i = 1, \ldots, n$ (v_i^* places every set of admissible controls of the player P $u_j(s_j)$, $s_j \in [\tau_{j-1}, \tau_j)$ $(j = 1, \ldots, i-1)$ in 1:1 correspondence with some value from *V*) such that $\pi z(t) \overline{\in} M$ holds for a trajectory $z(\tau)$ with origin at z_0, corresponding to any admissible control $u(\tau)$ of the player *P* and control $v(\tau) \equiv v_i^*(u_1(\cdot), \ldots, u_{i-1}(\cdot))$, $\tau \in [\tau_{i-1}, \tau_i)$ $(u_j(\tau) = u(\tau), \tau \in [\tau_{j-1}, \tau_j))$ of the player *E*.

Proof. Let $z_0 \overline{\in} P_\omega^* M$. Then there exists $v_1^* \in V$, such that for any $u_1 \in U$

$$\pi e^{At} z_0 + C_1 B(u_1, v_1^*) \overline{\in} M_2$$

From the results of /10/ it follows that since $B(U, v_1^*) - H^{\omega}$ is a convex set, therefore

$$\pi e^{At} z_0 + \int_{\tau_0}^{\tau_1} C(\tau) B(u_1(\tau), v_1^*)\, d\tau \overline{\in} M_2$$

for any admissible $u_1(\tau)$, $\tau \in [\tau_0, \tau_1)$.

Let us assume that the mappings v_i^*, $i < k$ have been constructed, and

$$x_{k-1} = \pi e^{At} z_0 + \sum_{i=1}^{k-1} \int_{\tau_i}^{\tau_{i-1}} C(\tau) B(u_i(\tau), v_i^*(u_1(\cdot), \ldots, u_{i-1}(\cdot)))\, d\tau \in M_k$$

The quantity x_{k-1} depends on $u_1(\cdot), \ldots, u_{k-1}(\cdot)$. Just as before, this, together with /10/, implies that $v_k^*(u_1(\cdot), \ldots, u_{k-1}(\cdot))$ exists such that for any admissible $u_k(\tau)$, $\tau \in [\tau_{k-1}, \tau_k]$

$$x_{k-1} + \int_{\tau_{k-1}}^{\tau_k} C(\tau) B(u_k(\tau), v_k^*(v_1(\cdot), \ldots, u_{k-1}(\cdot)))\, d\tau \in M_{k+1}$$

Continuing the process of constructing v_k^*, we obtain $\pi z(t) \in M = M_k$.

Using the strategies described above, we can construct ε-strategies of the players /4, 5/. Therefore the following corollary holds.

Corollary. Let M and $B(U, v)$ $(v \in V) - H^\omega$ be convex sets. Then $P_\omega^* M = P^t M_L$, where $P^t M_L$ is a set of all points from which the player P can terminate the game in his favour using the ε-strategies /4, 5/.

3. We will illustrate the results obtained by solving a differential game whose dynamics are described by the linear-pendulum equation.

Let the game be described by the equations

$$x^{\cdot} = y,\ y^{\cdot} = -Dx + B(u, v),\ x \in L \tag{3.1}$$

where D is a constant matrix. In this case $z = L \times L$ and the matrices A, π and φ can be represented in the form

$$A = \begin{Vmatrix} 0 & E \\ -D & 0 \end{Vmatrix},\quad \pi = \| E\ 0 \|,\quad \varphi = \begin{Vmatrix} 0 \\ E \end{Vmatrix}$$

It can be shown that

$$\pi e^{A\tau} = \| \cos(\sqrt{D}\,\tau)\ (\sqrt{D})^{-1} \sin(\sqrt{D}\tau) \| \pi e^{A\tau}\varphi = (\sqrt{D})^{-1} \sin(\sqrt{D}\tau)$$

where the formal notation is used for the series

$$\begin{aligned} \cos(\sqrt{D}\tau) &= E - \frac{1}{2!} D\tau^2 + \frac{1}{4!} D^2\tau^4 - \ldots \\ (\sqrt{D})^{-1} \sin(\sqrt{D}\tau) &= \tau - \frac{1}{3!} D\tau^3 + \frac{1}{5!} D^2\tau^5 - \ldots \end{aligned} \tag{3.2}$$

From this it follows that if $x(0) = x_0$, $x^{\cdot}(0) = y_0$, then the solution of system (3.1) can be written in the form

$$x(t) = \pi z(t) = \cos(\sqrt{D}\,t)\, x_0 + (\sqrt{D})^{-1} \sin(\sqrt{D}\,t)\, y_0 + \int_0^t (\sqrt{D})^{-1} \sin(\sqrt{D}\,(t-\tau))\, B(u(\tau), v(\tau))\, d\tau$$

Let λ_j $(j = 1, \ldots, k)$ be the eigenvalues of the operator D^*. We shall allow the possibility that the eigenvalues may include zero. We choose a decomposition $\omega = \{\tau_0 = 0 < \tau_1 < \ldots < \tau_n = t\}$, such that any function $y_j(\tau) = (\sqrt{\lambda_j})^{-1} \sin(\sqrt{\lambda_j}(t-\tau))$ will not change its sign in every interval $[\tau_{i-1}, \tau_i)$ $(i = 1, \ldots, n)$. Here, as above, we shall assume that $(\sqrt{\lambda_j})^{-1} \sin(\sqrt{\lambda_j}\,\tau)$ represents the formal notation of the second series of (3.2), provided that we replace D by λ_i in the latter. We can take as H^ω the set of all eigenvalues of the operator D^*. Indeed, if $x^* \in H^\omega$ corresponds to the eigenvalue λ_j, then $(\pi e^{A(t-\tau)}\varphi)^* x^* = y_j(\tau)\, x^*$, and conditions a) and b) of the definition of the set H^ω will hold.

In order to apply Theorem 2, it is sufficient to confirm the condition of existence of C_i^{-1}, where

$$C_i = \int_{\tau_{i-1}}^{\tau_i} (\sqrt{D})^{-1} \sin(\sqrt{D}\,(t-\tau))\, d\tau$$

This condition holds, in particular, when the operator D can be represented by a diagonal matrix.

REFERENCES

1. PONTRYAGIN L.S., Linear differential games of pursuit. Mat. Sb., 112, 3, 1980.
2. KRASOVSKII N.N. and SUBBOTIN A.I., Positional Differential Games. Nauka, Moscow, 1974.
3. KRASOVSKII N.N., Control of a Dynamic System. Nauka, Moscow, 1985.

4. PSHENICHNYI B.N., Structure of differential games. Dokl. Akad. Nauk SSSR, 184, 2, 1969.
5. PSHENICHNYI B.N. and SAGAIDAK M.I., On differential games with fixed time. Kibernetika, 2, 1970.
6. PSHENICHNYI B.N., CHIKRII A.A. and RAPPOPORT I.S. An effective method of solving differential games with numerous pursuers. Dokl. Akad. Nauk SSSR, 256, 3, 1981.
7. NIKOL'SKII M.S., On a straightforward method of solving linear differential games of pursuit-evasion. Mat. Zametki, 33, 6, 1983.
8. GUSYATNIKOV P.B. and POLOVINKIN E.S., On two types of the operators for biparametric equations. Dokl. Akad. Nauk SSSR, 254, 1, 1980.
9. OSTAPENKO V.V., Linear differential games in which the basic operations admit of a simple representation. Dokl. Akad. Nauk SSSR, 261, 4, 1981.
10. OSTAPENKO V.V., The method of H-convex sets in differential games. Dokl. Akad. Nauk UkSSR, 12, 1984.
11. BOLTYANSKII V.G. and SOLTAN P.S., Combinatorial geometry and classes of convexity. Uspekhi Mat. Nauk. 33, 1, 1978.

Translated by L.K.

PMM U.S.S.R.,Vol.51,No.3,pp.307-313,1987
Printed in Great Britain

0021-8928/87 $10.00+0.00

ON THE STABILIZATION OF CERTAIN NON-LINEAR SYSTEMS*

V.B. KOLMANOVSKII

The problem of the stabilization of systems with a non-linearity, dependent on a small parameter ε, is studied. A quasi-optimal stabilization algorithm is proposed and substantiated for the case of small ε. If nothing is known regarding the magnitude of ε, the technique of adaptive stabilization is developed. Examples of synthesis in the control of the motion of robots with unknown parameters are considered.

The problem of the stabilization of motions, with which a large number of investigations have been concerned, is studied in two formulations /1/. The first is associated with the determination of the control under which the system becomes stable while the second is associated with the choice of the control which minimizes a functional (the quality criterion). Generally speaking, the above-mentioned formulations are not equivalent. In order that the control which minimizes the integral quality criterion should simultaneously make the system stable, it follows that one should consider quality criteria which are positive-definite with respect to the phase coordinates (quadratic criteria, for example). Moreover, if the perturbations in the system are small, then a non-linear system may be approximated by a linear system. In fact, in the case of problems involving the stabilization of linear systems, the final results have been obtained with a quadratic quality criterion.

The question of whether it is possible to expand the Bellman function and the optimal control in power series in small perturbations and the convergence of these series has been investigated in /3, 4/ for non-linear problems and small perturbations. At the same time, the initial perturbations may not be small when real systems are treated and it is therefore necessary to take account of non-linearity when constructing the control.

A method of quasi-optimal stabilization is presented below and error estimates are obtained for the case of arbitrary initial perturbations.

1. Formulation of the quasi-optimal stabilization problem. A control system has the form

$$x^{\cdot}(t) = \varepsilon f(t, x(t)) + B(t)u, \ t \geqslant 0, \ x(0) = x_0 \tag{1.1}$$

*Prikl.Matem.Mekhan.,51,3,395-402,1987